Recall from last talk

Projections on convex subgraphs Y

X median graph.

$x \in V(X)$, $\underset{\text{sb-grph}}{Y \subseteq X}$ convex.

convex: for any $x, y \in Y$, all geodesics between x and y in X lie in Y.

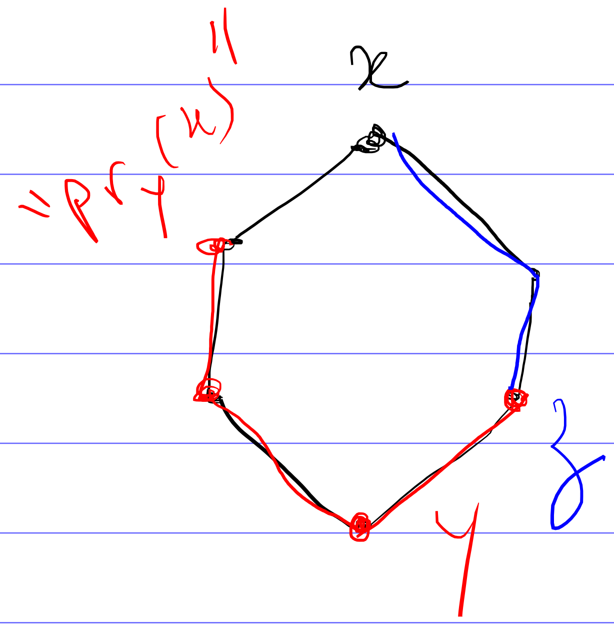
Define the projection of x on Y, denoted by $pr_Y(x)$, as the unique point in Y minimizing the distance from x to Y.

Definition - Theorem

Moreover, for every $z \in Y$, there exists a geodesic $[x, z]$ from x to z passing through $pr_Y(x)$.

Side note:

$X = C_6$



Lemma:

X median graph.

$x \in X$.

$Y \subseteq X$ convex subgraph.

Then, any hyperplane in X separating x from its projection $pr_Y(x)$, separates x and Y, and the converse trivially holds.

Property 1:

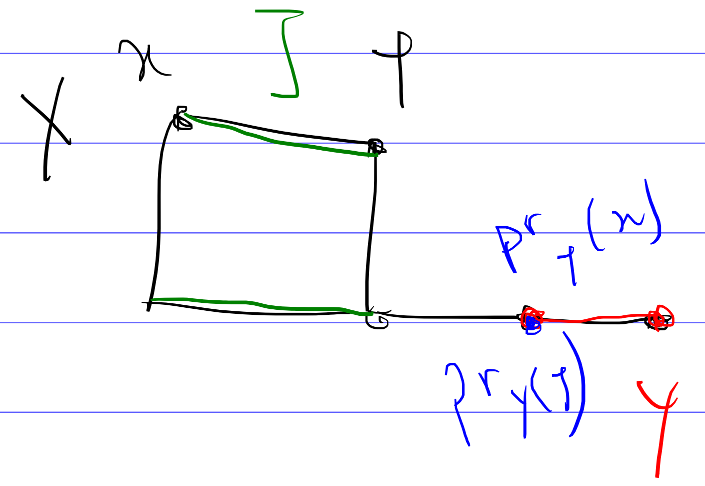
X ~~median~~ graph.

$Y \subseteq X$ convex subgraph.

$x, y \in X$.

A hyperplane $J \in \mathcal{H}(X)$ separates $pr_Y(x)$ and $pr_Y(y)$ iff J crosses Y and separates x and y.

Counter-eg.

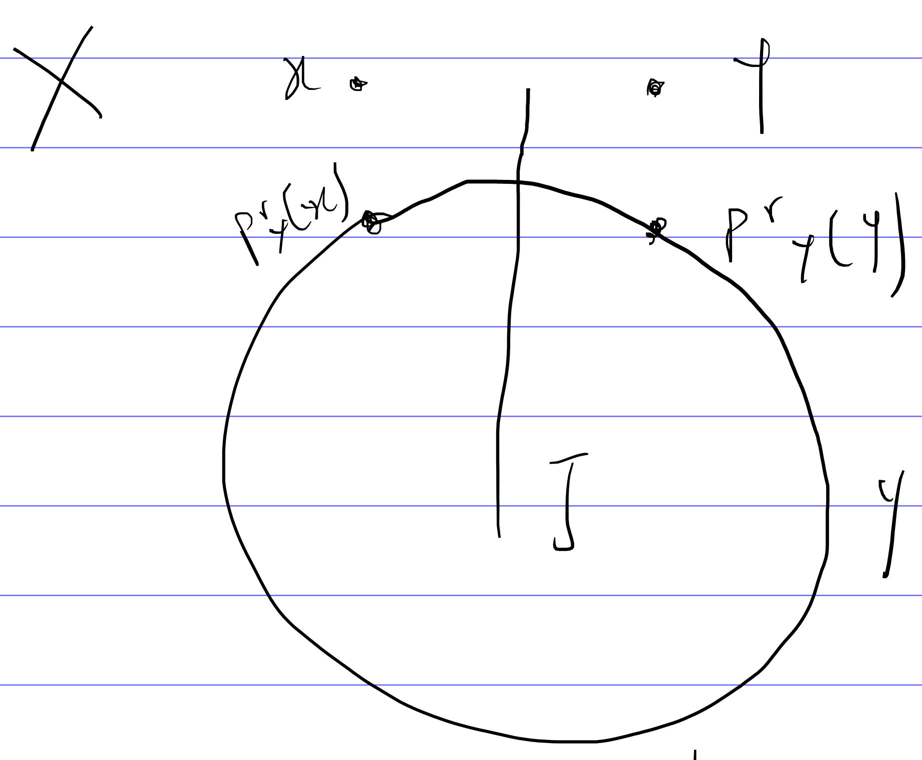


Proof of the above property

($\Longrightarrow$):

$J \in \mathcal{H}(X)$.

J separates $pr_Y(x)$ and $pr_Y(y)$



Since J separates $pr_Y(x)$ and $pr_Y(y)$, J crosses Y.

Now, since J crosses Y, J then doesn't separate

x from Y, neither y from Y; therefore,

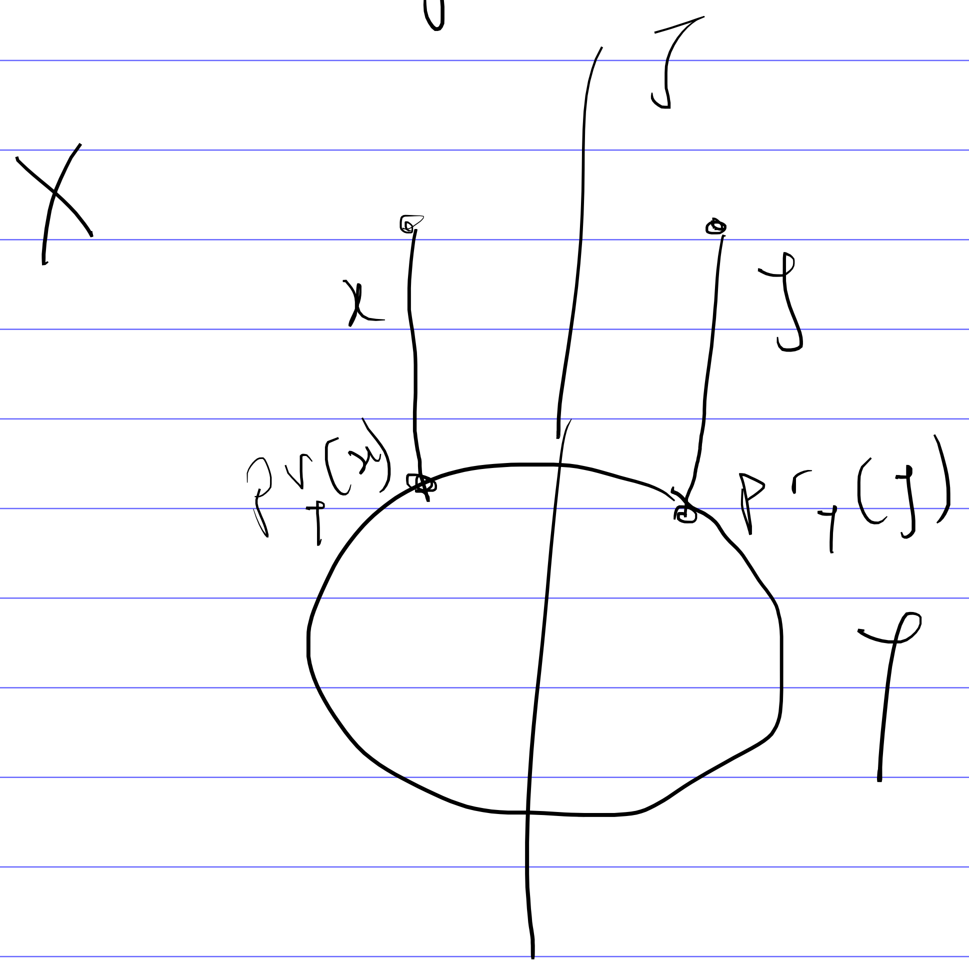
J doesn't separate x from $pr_Y(x)$ on the one hand, on the other hand, J doesn't separate y from $pr_Y(y)$.

This yields J separating x and y.

($\Leftarrow$):

$J \in \mathcal{H}(X)$

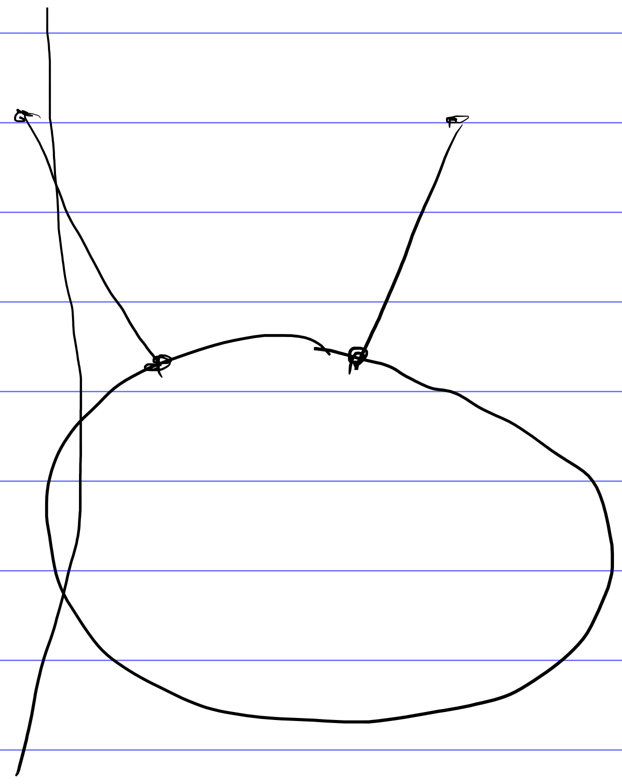
J separates x and y, and J crosses Y.



Since J crosses Y, J doesn't separate x from Y, neither y from Y. By the above lemma (contraposition), J doesn't separate x from $pr_Y(x)$, neither J separates y from $pr_Y(y)$.

We conclude that J separates $pr_Y(x)$ and $pr_Y(y)$.

$\square$



Property 2:

Projection on convex subgraphs is a retraction.

Proof:

Let X be a median graph, $Y \subseteq X$ a convex subgraph, and $x, y \in V(X)$.

To show.

$$\boxed{d\left(pr_Y(x), pr_Y(y)\right) \leq d(x, y)}$$

$$\begin{aligned}
d(pr_Y(x), pr_Y(y)) &= \#\{J \in \mathcal{H}(X) \mid J \text{ separates } pr_Y(x) \text{ and } pr_Y(y)\} \\
&= \#\{J \in \mathcal{H}(X) \mid J \text{ separates } x \text{ and } y \text{ AND } J \text{ crosses } Y\} \\
&\leq \#\{J \in \mathcal{H}(X) \mid J \text{ separates } x \text{ and } y\} \\
&= d(x, y).
\end{aligned}$$

$\square$

Big theorem:

(Interaction between hyperplanes and projections in median graphs)

Theorem:

Let X be a median graph, and let Y and Z be two convex subgraphs of X. The following hold:

(i) $pr_Y(Z)$ and $pr_Z(Y)$ are convex.

(ii) For any $y \in Y$, y and $pr_Z(y)$ minimize the distance between Y and Z.

(iii) For any vertices $y \in Y$ and $z \in Z$ minimizing the distance between Y and Z, the hyperplanes separating y and z coincide with the hyperplanes separating Y and Z.

(iv) A hyperplane crosses $P_Z(Y)$ iff it crosses $P_Y(Z)$ iff it crosses both Y and Z.

Convex hulls of subgraphs of median graphs

Take a subgraph Y of a median graph X and "make it" convex so we can project onto it.

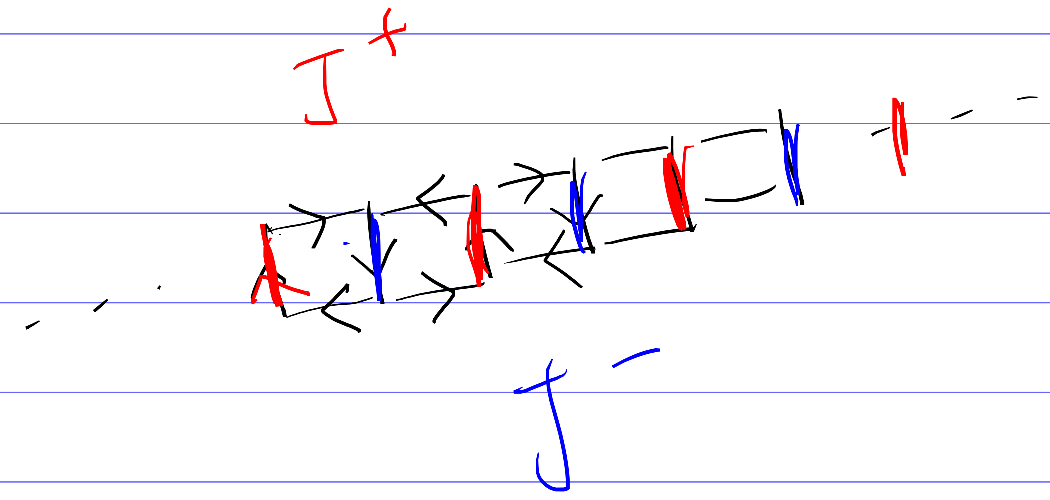
Definition:

X median graph.

$Y \subseteq X$ subgraph of X.

Define the convex hull of Y, denoted by $\mathrm{conv}(Y)$ to be:

$$\mathrm{conv}(Y) := \bigcap_{\substack{Z \subseteq X \text{ subgraph} \\ Z \supseteq Y \\ Z \text{ convex}}} Z = \text{smallest convex subgraph of } X \text{ containing } Y.$$



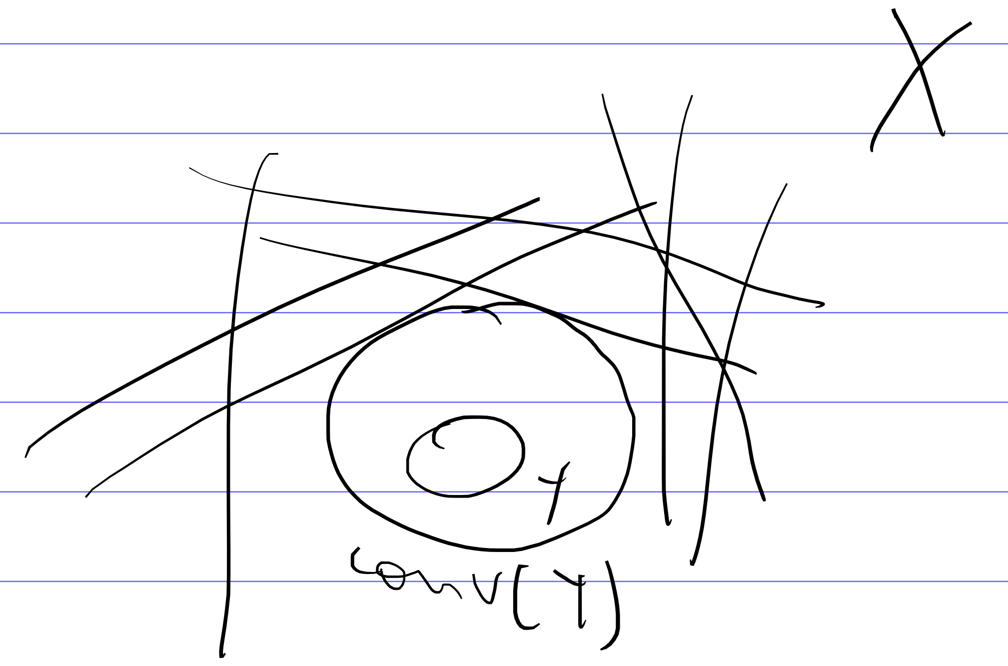
Theorem: (important)

X median graph.

$Y \subseteq X$ a connected subgraph of X.

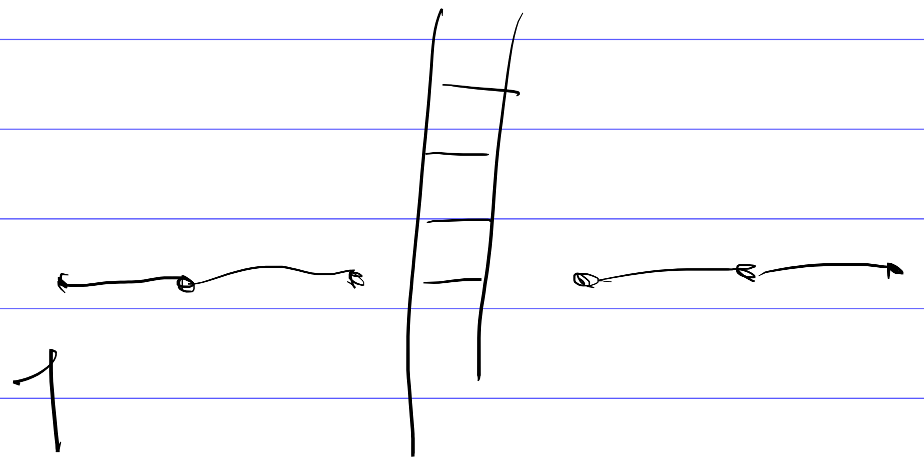
We have:

(i) $\mathrm{conv}(Y) = \bigcap_{\substack{J \in \mathcal{H}(X) \\ J^+ \supseteq Y}} J^+$.



(ii) For every hyperplane J of X, J crosses Y iff J crosses $\mathrm{conv}(Y)$.

(iii) For every two hyperplanes J and K of X crossing $\mathrm{conv}(Y)$, J and K are transverse in X iff they are transverse in $\mathrm{conv}(Y)$.



Intervals between two points in a median graph

Let X be a median graph.

$x, y \in V(X)$.

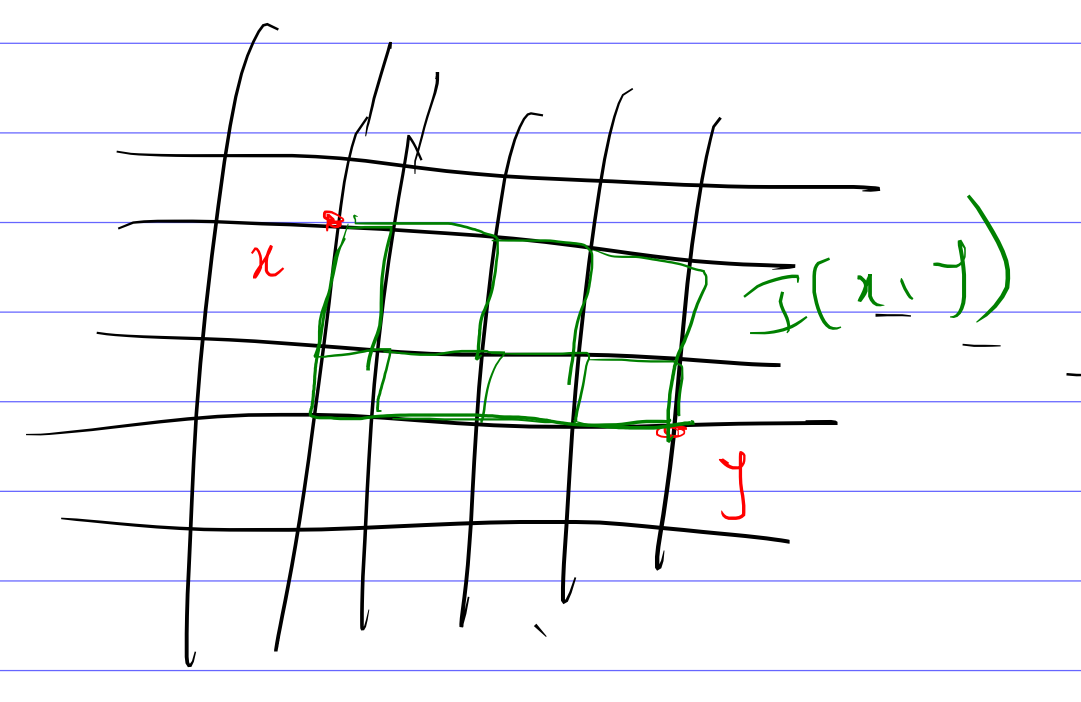
Define the interval between x and y, denoted by $I(x,y)$, to be the following subgraph of X:

$$I(x,y) := \{ z \in V(X) \mid d(x,y) = d(x,z) + d(z,y) \}$$

$$= \{ \text{geodesics between } x \text{ and } y \}.$$

$x, y \in I(x,y)$.

e.g.: in $\mathbb{H}^2$:



Property:

Intervals are convex.

Proof:

The idea is to show that $I(x,y) = \mathrm{conv}(I(x,y))$.

$(\subseteq)$: $I(x,y) \subseteq \mathrm{conv}(I(x,y))$ clear.

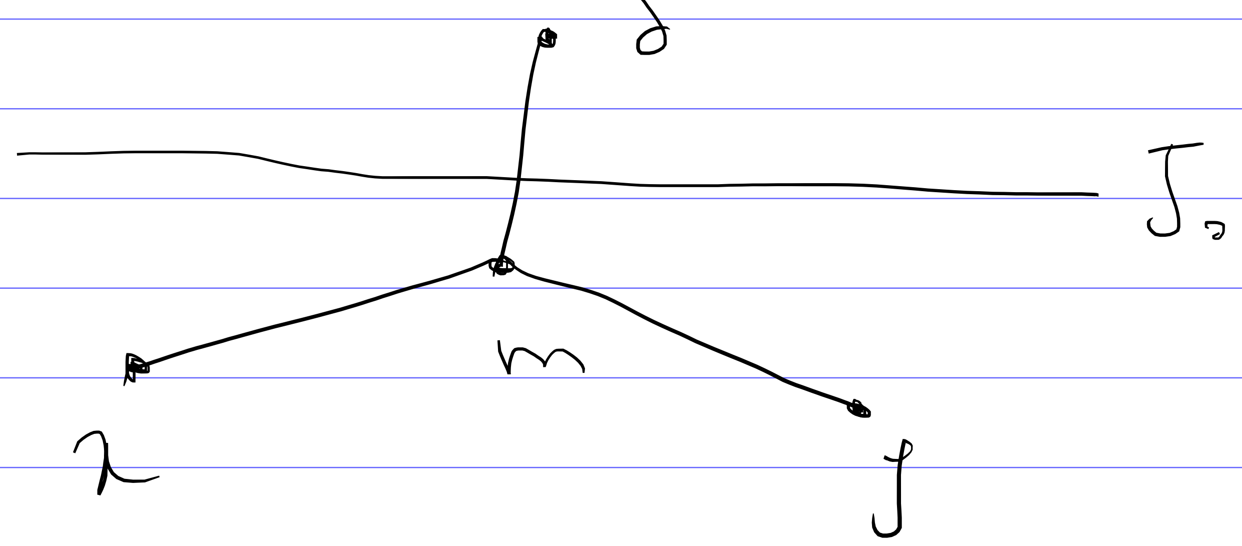
$(\supseteq)$:

$\forall z \notin I(x,y),\ z \notin \mathrm{conv}(I(x,y))$.

Let $z \notin I(x,y)$.

Let $m := \mu(x,y,z)$.

m lies on a geodesic between x and y, so $z \neq m$.



$z \in J_0^+$ So there exists a hyperplane $J_0 \in \mathcal{H}(X)$ separating

$m \in J_0^-$ z and m, so crossing exactly once each of the geodesics

$[x,m] \cup [m,z]$ and $[y,m] \cup [m,z]$.

$\Rightarrow z \in J_0^+$ and $x,y \in J_0^-$

convex

$\Rightarrow$ every geodesic between x and y lies fully in J_0^-

$\Rightarrow I(x,y) \subseteq J_0^-$

So $z \notin J_0^-$ containing $I(x,y)$

$$\Rightarrow z \notin \bigcap_{\substack{J \in \mathcal{H}(X) \\ J^- \supseteq I(x,y)}} J^-$$

$\Rightarrow z \notin \mathrm{conv}(I(x, y))$.

$\therefore$ $I(x, y)$ is convex. $\square$